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State-Space Averaging Technique

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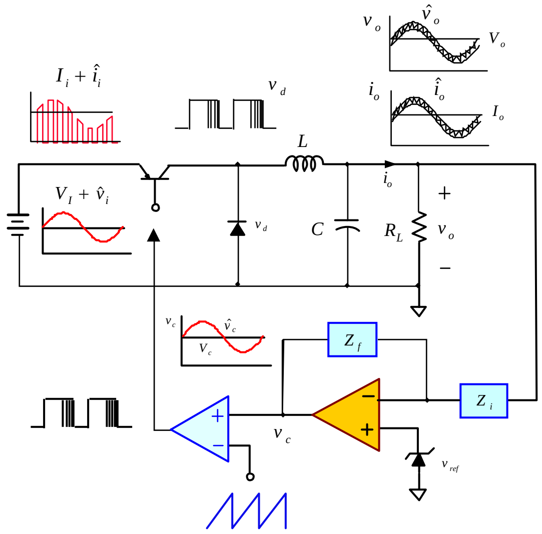
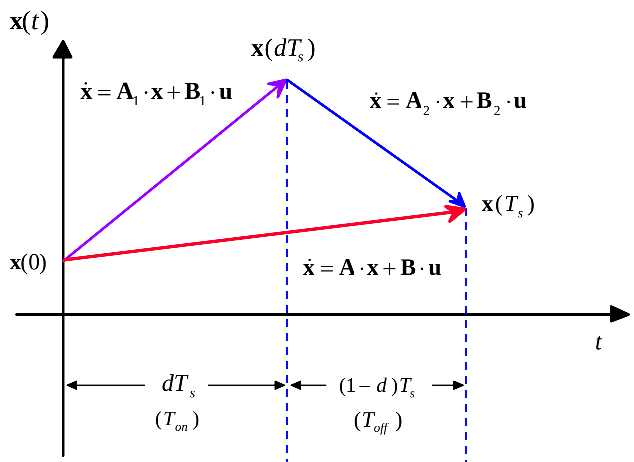
State-Space Averaging Techniques

- State Space Averaging Modeling
- Static Analysis
- Small-Signal Model at CCM
- Small-Signal Model at DCM
- Frequency Response Analysis

Concept of Time Averaging (Low Frequency Response Behavior)



Linear Approximation of State Space Trajectories



State-Space Averaging Method (CCM)

During T_{ON} $\begin{cases} \dot{\mathbf{x}} = \mathbf{A}_1\mathbf{x} + \mathbf{B}_1\mathbf{u} \\ \mathbf{y} = \mathbf{C}_1\mathbf{x} + \mathbf{E}_1\mathbf{u} \end{cases}$

During T_{OFF} $\begin{cases} \dot{\mathbf{x}} = \mathbf{A}_2\mathbf{x} + \mathbf{B}_2\mathbf{u} \\ \mathbf{y} = \mathbf{C}_2\mathbf{x} + \mathbf{E}_2\mathbf{u} \end{cases}$

When the circuit time constant is far greater than the switching period, the above equations can be averaged as:

Switch-ON Period | Switch-OFF Period

$$\boxed{\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}_1\mathbf{x} + \mathbf{B}_1\mathbf{u} \\ \mathbf{y} &= \mathbf{C}_1\mathbf{x} + \mathbf{E}_1\mathbf{u} \end{aligned}} \times d_1 + \boxed{\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}_2\mathbf{x} + \mathbf{B}_2\mathbf{u} \\ \mathbf{y} &= \mathbf{C}_2\mathbf{x} + \mathbf{E}_2\mathbf{u} \end{aligned}} \times d_2$$

averaging by using state duty ratio weighting

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State-Space Averaging Method

$$\dot{\mathbf{x}} = (\mathbf{A}_1 d_1 + \mathbf{A}_2 d_2)\mathbf{x} + (\mathbf{B}_1 d_1 + \mathbf{B}_2 d_2)\mathbf{u}$$
$$\mathbf{y} = (\mathbf{C}_1 d_1 + \mathbf{C}_2 d_2)\mathbf{x} + (\mathbf{E}_1 d_1 + \mathbf{E}_2 d_2)\mathbf{u}$$

$$\mathbf{x} = \mathbf{Ax} + \mathbf{Bu}$$
$$\mathbf{y} = \mathbf{Cx} + \mathbf{Du}$$

where

$$\mathbf{A} = \mathbf{A}_1 d_1 + \mathbf{A}_2 d_2$$
$$\mathbf{B} = \mathbf{B}_1 d_1 + \mathbf{B}_2 d_2$$
$$\mathbf{C} = \mathbf{C}_1 d_1 + \mathbf{C}_2 d_2$$
$$\mathbf{E} = \mathbf{E}_1 d_1 + \mathbf{E}_2 d_2$$

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State Averaging by Active Duty Ratios

During T_{ON} $\begin{cases} \dot{\mathbf{x}} = \mathbf{A}_1\mathbf{x} + \mathbf{B}_1\mathbf{u} \\ \mathbf{y} = \mathbf{C}_1\mathbf{x} + \mathbf{E}_1\mathbf{u} \end{cases}$

During T_{OFF} $\begin{cases} \dot{\mathbf{x}} = \mathbf{A}_2\mathbf{x} + \mathbf{B}_2\mathbf{u} \\ \mathbf{y} = \mathbf{C}_2\mathbf{x} + \mathbf{E}_2\mathbf{u} \end{cases}$

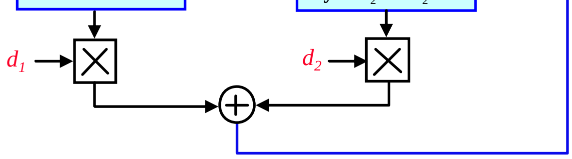
When the circuit time constant is far greater than the switching period, the above equations can be averaged as:

Switch-ON Period

$\dot{\mathbf{x}} = \mathbf{A}_1\mathbf{x} + \mathbf{B}_1\mathbf{u}$
$\mathbf{y} = \mathbf{C}_1\mathbf{x} + \mathbf{E}_1\mathbf{u}$

Switch-OFF Period

$\dot{\mathbf{x}} = \mathbf{A}_2\mathbf{x} + \mathbf{B}_2\mathbf{u}$
$\mathbf{y} = \mathbf{C}_2\mathbf{x} + \mathbf{E}_2\mathbf{u}$



$$\dot{\mathbf{x}} = (\mathbf{A}_1 d_1 + \mathbf{A}_2 d_2)\mathbf{x} + (\mathbf{B}_1 d_1 + \mathbf{B}_2 d_2)\mathbf{u}$$
$$\mathbf{y} = (\mathbf{C}_1 d_1 + \mathbf{C}_2 d_2)\mathbf{x} + (\mathbf{E}_1 d_1 + \mathbf{E}_2 d_2)\mathbf{u}$$

$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$
$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$
where
$\mathbf{A} = \mathbf{A}_1 d_1 + \mathbf{A}_2 d_2$
$\mathbf{B} = \mathbf{B}_1 d_1 + \mathbf{B}_2 d_2$
$\mathbf{C} = \mathbf{C}_1 d_1 + \mathbf{C}_2 d_2$
$\mathbf{E} = \mathbf{E}_1 d_1 + \mathbf{E}_2 d_2$

state averaging by using duty ratio weighting

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Small-Signal Perturbation at a DC Operating Point

Substitute $\mathbf{x} = \mathbf{X} + \hat{\mathbf{x}}$; $\mathbf{y} = \mathbf{Y} + \hat{\mathbf{y}}$; $\mathbf{u} = \mathbf{U} + \hat{\mathbf{u}}$;
$d_1 = D_1 + \hat{d}$; $d_2 = D_2 - \hat{d}$

$$\frac{d}{dt}(\mathbf{X}+\hat{\mathbf{x}}) = \left[\mathbf{A}_1(D_1+\hat{d}) + \mathbf{A}_2(D_2-\hat{d})\right](\mathbf{X}+\hat{\mathbf{x}}) + \left[\mathbf{B}_1(D_1+\hat{d}) + \mathbf{B}_2(D_2-\hat{d})\right](\mathbf{U}+\hat{\mathbf{u}})$$

$$\overbrace{\frac{d}{dt}\mathbf{X}}^{\text{dc terms}=0} + \frac{d}{dt}\hat{\mathbf{x}} = \overbrace{(\mathbf{A}_1 D_1 + \mathbf{A}_2 D_2)\mathbf{X} + (\mathbf{B}_1 D_1 + \mathbf{B}_2 D_2)\mathbf{U}_i}^{\text{dc terms}=0} +$$
$$(\mathbf{A}_1 D_1 + \mathbf{A}_2 D_2)\hat{\mathbf{x}} + (\mathbf{B}_1 D_1 + \mathbf{B}_2 D_2)\hat{\mathbf{u}} +$$
$$\left[(\mathbf{A}_1 - \mathbf{A}_2)\mathbf{X} + (\mathbf{B}_1 - \mathbf{B}_2)\mathbf{U}\right]\hat{d} +$$
$$\underbrace{(\mathbf{A}_1 - \mathbf{A}_2)\hat{d}\hat{\mathbf{x}} + (\mathbf{B}_1 - \mathbf{B}_2)\hat{d}\hat{\mathbf{u}}}_{\text{ignore nonlinear term}}$$

Note:
The nonlinear dynamic system is linearized around a selected operating point!

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DC Model and AC Model

DC Model

$$\mathbf{X} = -\mathbf{A}^{-1}\mathbf{B}\mathbf{U}$$

$$\mathbf{Y} = (-\mathbf{C}\mathbf{A}^{-1}\mathbf{B} + \mathbf{E})\mathbf{U}$$

AC Model

$$\frac{d}{dt}\hat{\mathbf{x}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\hat{\mathbf{u}} + \mathbf{F}\hat{d}$$

$$\hat{\mathbf{y}} = \mathbf{C}\hat{\mathbf{x}} + \mathbf{E}\hat{\mathbf{u}} + \mathbf{G}\hat{d}$$

where

$$\mathbf{A} = \mathbf{A}_1 D_1 + \mathbf{A}_2 D_2$$
$$\mathbf{B} = \mathbf{B}_1 D_1 + \mathbf{B}_2 D_2$$
$$\mathbf{C} = \mathbf{C}_1 D_1 + \mathbf{C}_2 D_2$$
$$\mathbf{E} = \mathbf{E}_1 D_1 + \mathbf{E}_2 D_2$$
$$\mathbf{F} = (\mathbf{A}_1 - \mathbf{A}_2)\mathbf{X} + (\mathbf{B}_1 - \mathbf{B}_2)\mathbf{U}$$
$$\mathbf{G} = (\mathbf{C}_1 - \mathbf{C}_2)\mathbf{X} + (\mathbf{E}_1 - \mathbf{E}_2)\mathbf{U}$$

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Transfer Function Matrix

$$\mathbf{x}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{u}(s) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{F}d(s)$$
$$= \mathbf{H}_u(s)\mathbf{u}(s) + \mathbf{H}_d(s)d(s)$$
$$\mathbf{H}_u(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$
$$\mathbf{H}_d(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{F}d(s)$$

$$\mathbf{y}(s) = [\mathbf{C}(sI - \mathbf{A})^{-1}\mathbf{B} + \mathbf{E}]\mathbf{u}(s) + [\mathbf{C}(sI - \mathbf{A})^{-1}\mathbf{F} + \mathbf{G}]d(s)$$
$$= \mathbf{G}_u(s)\mathbf{u}(s) + \mathbf{G}_d(s)d(s)$$

$$\mathbf{G}_u(s) = \mathbf{C}(sI - \mathbf{A})^{-1}\mathbf{B} + \mathbf{E}$$

$$\mathbf{G}_d(s) = \mathbf{C}(sI - \mathbf{A})^{-1}\mathbf{F} + \mathbf{G}$$

$$\begin{bmatrix} v_o(s) \\ i_i(s) \end{bmatrix} = \begin{bmatrix} G_{u11}(s) & G_{u12}(s) \\ G_{u21}(s) & G_{u22}(s) \end{bmatrix} \begin{bmatrix} v_i(s) \\ i_d(s) \end{bmatrix} + \begin{bmatrix} G_{d1}(s) \\ G_{d2}(s) \end{bmatrix} d(s)$$

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Open Loop Transfer Functions

- **control-to-output (open-loop transfer function):**

$$\frac{v_o(s)}{d(s)} = G_{d1}(s) = \mathbf{G}_d(s)\big|_1 = \left[\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{F}+\mathbf{G}\right]\Big|_1$$

- **line-to-output (audio susceptibility):**

$$\frac{v_o(s)}{v_i(s)} = G_{u11}(s) = \mathbf{G}_u(s)\big|_{11} = \left[\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{E}\right]\Big|_{11}$$

- **output impedance:**

$$\frac{v_o(s)}{i_d(s)} = G_{u12}(s) = \mathbf{G}_u(s)\big|_{12} = \left[\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{E}\right]\Big|_{12}$$

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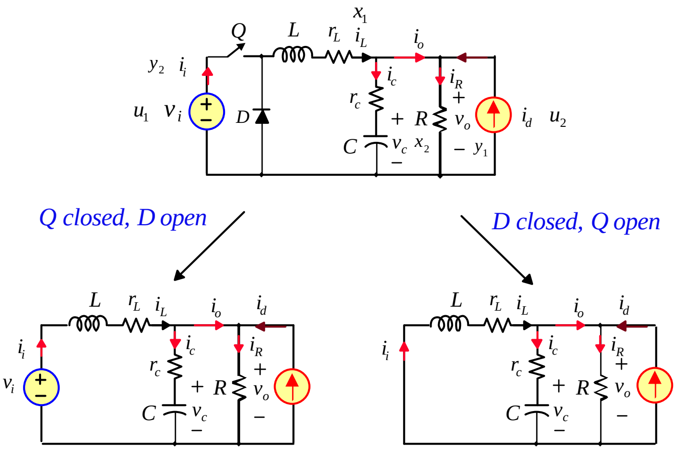
Example: Buck Converter

Step 1. Draw the linear equivalent circuit for each switching state of the converter.



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Select State Variables

Select the inductor current i_L and capacitor voltage v_L state variables.

$$\mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} i_L \\ v_c \end{bmatrix}$$

Select the input dc voltage and output disturbance current as input variables.

$$\mathbf{u}=\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}=\begin{bmatrix} v_i \\ i_d \end{bmatrix}$$

Select the output voltage and input current as output variables.

$$\mathbf{y}=\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}=\begin{bmatrix} v_o \\ i_i \end{bmatrix}$$

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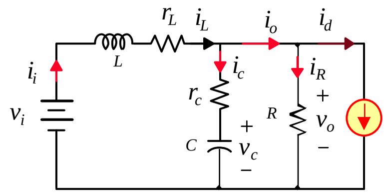
Derive Circuit Equations: State Equations

Step 2. Write the circuit equations for each equivalent circuit in a state-variable format.

Q-ON and D-OFF State:



❶ $$v_i = L\frac{di_L}{dt} + i_L r_L + r_C C\frac{dv_c}{dt} + v_C$$

❷ $$r_C C\frac{dv_c}{dt} + v_C = (i_L - i_C - i_d)R$$
$$= (i_L - C\frac{dv_c}{dt} + i_d)R$$
$$= i_L R - RC\frac{dv_c}{dt} + i_d R$$

$$(R+r_C)C\frac{dv_c}{dt} = i_L R - v_C - i_d R$$

⇩

$$\frac{dv_c}{dt} = \frac{R}{(R+r_C)C} i_L - \frac{1}{(R+r_C)C} v_C - \frac{R}{(R+r_C)C} i_d$$

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State-Space Average Modeling: State Equations

Replace $C\frac{dv_c}{dt}$ in ❶ with $\frac{R}{(R+r_C)}i_L-\frac{1}{(R+r_C)}v_C-\frac{R}{(R+r_C)}i_d$

We can obtain:

$$v_i=L\frac{di_L}{dt}+i_Lr_L+r_C\left(\frac{R}{(R+r_C)}i_L-\frac{1}{(R+r_C)}v_C-\frac{R}{(R+r_C)}i_d\right)+v_C$$

$$L\frac{di_L}{dt}=v_i-i_Lr_L-\frac{Rr_C}{(R+r_C)}i_L+\frac{r_C}{(R+r_C)}v_C-\frac{Rr_C}{(R+r_C)}i_d-v_C$$

$$=-\left(\frac{Rr_C}{R+r_C}+r_L\right)i_L-\left(\frac{R}{R+r_C}\right)v_C+v_i-\frac{Rr_C}{R+r_C}i_d$$

⇩

$$\frac{di_L}{dt}=-\frac{1}{L}\left(\frac{Rr_C}{R+r_C}+r_L\right)i_L-\frac{1}{L}\left(\frac{R}{R+r_C}\right)v_C+\frac{1}{L}v_i-\frac{1}{L}\left(\frac{Rr_C}{R+r_C}\right)i_d$$

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State-Space Average Modeling: State Equations

❸ $i_i=i_L$

❹
$$v_o=(i_L-i_C-i_d)R$$
$$=i_LR-i_CR-i_dR$$
$$=i_LR-RC\left(\frac{R}{(R+r_C)C}i_L-\frac{1}{(R+r_C)C}v_C-\frac{R}{(R+r_C)C}i_d\right)-i_dR$$
$$=\frac{Rr_C}{(R+r_C)}i_L+\frac{R}{(R+r_C)}v_C-\frac{Rr_C}{(R+r_C)}i_d$$

$$v_o=\frac{Rr_C}{(R+r_C)}i_L+\frac{R}{(R+r_C)}v_C-\frac{Rr_C}{(R+r_C)}i_d$$

$$\begin{bmatrix}\dot{i}_L\\ \dot{v}_C\end{bmatrix}=\begin{bmatrix}-\frac{1}{L}\left(\frac{Rr_C}{R+r_C}+r_L\right) & -\frac{1}{L}\left(\frac{R}{R+r_C}\right)\\ \frac{1}{C}\left(\frac{R}{R+r_C}\right) & -\frac{1}{C}\left(\frac{1}{R+r_C}\right)\end{bmatrix}\begin{bmatrix}i_L\\ v_C\end{bmatrix}+\begin{bmatrix}\frac{1}{L} & -\frac{1}{L}\left(\frac{Rr_C}{R+r_C}\right)\\ 0 & -\frac{1}{C}\left(\frac{R}{R+r_C}\right)\end{bmatrix}\begin{bmatrix}v_i\\ i_d\end{bmatrix}$$

$$\begin{bmatrix}v_o\\ i_i\end{bmatrix}=\begin{bmatrix}\frac{Rr_C}{R+r_C} & \frac{R}{R+r_C}\\ 1 & 0\end{bmatrix}\begin{bmatrix}i_L\\ v_C\end{bmatrix}+\begin{bmatrix}0 & -\frac{Rr_C}{R+r_C}\\ 0 & 0\end{bmatrix}\begin{bmatrix}v_i\\ i_d\end{bmatrix}$$

Q-OFF and D-ON State Equation

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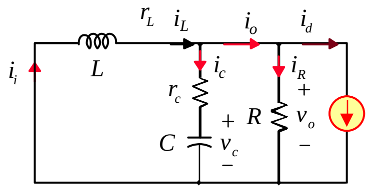
State-Space Average Modeling: State Equations

Q-OFF and D-ON State:



❶ $0 = L\frac{di_L}{dt} + i_L r_L + r_C C\frac{dv_c}{dt} + v_C$

❷ $r_C C\frac{dv_c}{dt} + v_C = (i_L - i_C - i_d)R$

$$= (i_L - C\frac{dv_c}{dt} - i_d)R$$

$$= i_L R - RC\frac{dv_c}{dt} - i_d R$$

$$(R+r_C)C\frac{dv_c}{dt} = i_L R - v_C - i_d R$$

⇩

$$\frac{dv_c}{dt} = \frac{R}{(R+r_C)C} i_L - \frac{1}{(R+r_C)C} v_C - \frac{R}{(R+r_C)C} i_d$$

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State-Space Average Modeling: State Equations

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$$(R+r_C)C\frac{dv_c}{dt} = i_L R - v_C - i_d R$$

⇩

$$\frac{dv_c}{dt} = \frac{R}{(R+r_C)C} i_L - \frac{1}{(R+r_C)C} v_C - \frac{R}{(R+r_C)C} i_d$$

Replace $C\frac{dv_c}{dt}$ in ❶ with $\frac{R}{(R+r_C)} i_L - \frac{1}{(R+r_C)} v_C - \frac{R}{(R+r_C)} i_d$

We can obtain: $0 = L\frac{di_L}{dt} + i_L r_L + r_C\left(\frac{R}{(R+r_C)} i_L - \frac{1}{(R+r_C)} v_C - \frac{R}{(R+r_C)} i_d\right) + v_C$

⇩

$$L\frac{di_L}{dt} = -i_L r_L - \frac{Rr_C}{(R+r_C)} i_L + \frac{r_C}{(R+r_C)} v_C - \frac{Rr_C}{(R+r_C)} i_d - v_C$$

$$= -\left(\frac{Rr_C}{R+r_C} + r_L\right) i_L - \left(\frac{R}{R+r_C}\right) v_C - \frac{Rr_C}{R+r_C} i_d$$

⇨

$$\frac{di_L}{dt} = -\frac{1}{L}\left(\frac{Rr_C}{R+r_C} + r_L\right) i_L - \frac{1}{L}\left(\frac{R}{R+r_C}\right) v_{Ci} - \frac{1}{L}\left(\frac{Rr_C}{R+r_C}\right) i_d$$

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State-Space Average Modeling: State Equations

❸ $i_i = 0$

❹
$$\begin{aligned} v_o &= (i_L - i_C - i_d)R \\ &= i_L R - i_C R - i_d R \\ &= i_L R - RC\left(\frac{R}{(R+r_C)C} i_L - \frac{1}{(R+r_C)C} v_C - \frac{R}{(R+r_C)C} i_d\right) - i_d R \\ &= \frac{Rr_C}{(R+r_C)} i_L + \frac{R}{(R+r_C)} v_C - \frac{Rr_C}{(R+r_C)} i_d \end{aligned}$$

$$v_o = \frac{Rr_C}{(R+r_C)} i_L + \frac{R}{(R+r_C)} v_C - \frac{Rr_C}{(R+r_C)} i_d$$

$$\begin{bmatrix} \dot{i}_L \\ \dot{v}_C \end{bmatrix} = \begin{bmatrix} -\frac{1}{L}\left(\frac{Rr_C}{R+r_C} + r_L\right) & -\frac{1}{L}\left(\frac{R}{R+r_C}\right) \\ \frac{1}{C}\left(\frac{R}{R+r_C}\right) & -\frac{1}{C}\left(\frac{1}{R+r_C}\right) \end{bmatrix} \begin{bmatrix} i_L \\ v_C \end{bmatrix} + \begin{bmatrix} 0 & -\frac{1}{L}\left(\frac{Rr_C}{R+r_C}\right) \\ 0 & -\frac{1}{C}\left(\frac{R}{R+r_C}\right) \end{bmatrix} \begin{bmatrix} v_i \\ i_d \end{bmatrix}$$

$$\begin{bmatrix} v_o \\ i_i \end{bmatrix} = \begin{bmatrix} \frac{Rr_C}{R+r_C} & \frac{R}{R+r_C} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} i_L \\ v_C \end{bmatrix} + \begin{bmatrix} 0 & -\frac{Rr_C}{R+r_C} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_i \\ i_d \end{bmatrix}$$

Q-ON and D-OFF State Equation

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State-Space Average Modeling: State Equations

Q Conducted:

$$\begin{bmatrix} \dot{i}_L \\ \dot{v}_C \end{bmatrix} = \begin{bmatrix} -\frac{1}{L}\left(\frac{Rr_C}{R+r_C} + r_L\right) & -\frac{1}{L}\left(\frac{R}{R+r_C}\right) \\ \frac{1}{C}\left(\frac{R}{R+r_C}\right) & -\frac{1}{C}\left(\frac{1}{R+r_C}\right) \end{bmatrix} \begin{bmatrix} i_L \\ v_C \end{bmatrix} + \begin{bmatrix} \frac{1}{L} & -\frac{1}{L}\left(\frac{Rr_C}{R+r_C}\right) \\ 0 & -\frac{1}{C}\left(\frac{R}{R+r_C}\right) \end{bmatrix} \begin{bmatrix} v_i \\ i_d \end{bmatrix}$$

$$\begin{bmatrix} v_o \\ i_i \end{bmatrix} = \begin{bmatrix} \frac{Rr_C}{R+r_C} & \frac{R}{R+r_C} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} i_L \\ v_C \end{bmatrix} + \begin{bmatrix} 0 & -\frac{Rr_C}{R+r_C} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_i \\ i_d \end{bmatrix}$$

D Conducted:

$$\begin{bmatrix} \dot{i}_L \\ \dot{v}_C \end{bmatrix} = \begin{bmatrix} -\frac{1}{L}\left(\frac{Rr_C}{R+r_C} + r_L\right) & -\frac{1}{L}\left(\frac{R}{R+r_C}\right) \\ \frac{1}{C}\left(\frac{R}{R+r_C}\right) & -\frac{1}{C}\left(\frac{1}{R+r_C}\right) \end{bmatrix} \begin{bmatrix} i_L \\ v_C \end{bmatrix} + \begin{bmatrix} 0 & -\frac{1}{L}\left(\frac{Rr_C}{R+r_C}\right) \\ 0 & -\frac{1}{C}\left(\frac{R}{R+r_C}\right) \end{bmatrix} \begin{bmatrix} v_i \\ i_d \end{bmatrix}$$

$$\begin{bmatrix} v_o \\ i_i \end{bmatrix} = \begin{bmatrix} \frac{Rr_C}{R+r_C} & \frac{R}{R+r_C} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} i_L \\ v_C \end{bmatrix} + \begin{bmatrix} 0 & -\frac{Rr_C}{R+r_C} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_i \\ i_d \end{bmatrix}$$

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State-Space Average Modeling: State Equations

Q Conducted:

$$\mathbf{A}_1 = \begin{bmatrix} -\frac{1}{L}\left(\frac{Rr_C}{R+r_C}+r_L\right) & -\frac{1}{L}\left(\frac{R}{R+r_C}\right) \\ \frac{1}{C}\left(\frac{R}{R+r_C}\right) & -\frac{1}{C}\left(\frac{1}{R+r_C}\right) \end{bmatrix}, \quad \mathbf{B}_1 = \begin{bmatrix} \frac{1}{L} & -\frac{1}{L}\left(\frac{Rr_C}{R+r_C}\right) \\ 0 & -\frac{1}{C}\left(\frac{R}{R+r_C}\right) \end{bmatrix}$$

$$\mathbf{C}_1 = \begin{bmatrix} \frac{Rr_C}{R+r_C} & \frac{R}{R+r_C} \\ 1 & 0 \end{bmatrix}, \quad \mathbf{E}_1 = \begin{bmatrix} 0 & -\frac{Rr_C}{R+r_C} \\ 0 & 0 \end{bmatrix}$$

D Conducted:

$$\mathbf{A}_2 = \begin{bmatrix} -\frac{1}{L}\left(\frac{Rr_C}{R+r_C}+r_L\right) & -\frac{1}{L}\left(\frac{R}{R+r_C}\right) \\ \frac{1}{C}\left(\frac{R}{R+r_C}\right) & -\frac{1}{C}\left(\frac{1}{R+r_C}\right) \end{bmatrix}, \quad \mathbf{B}_2 = \begin{bmatrix} 0 & -\frac{1}{L}\left(\frac{Rr_C}{R+r_C}\right) \\ 0 & -\frac{1}{C}\left(\frac{R}{R+r_C}\right) \end{bmatrix}$$

$$\mathbf{C}_2 = \begin{bmatrix} \frac{Rr_C}{R+r_C} & \frac{R}{R+r_C} \\ 0 & 0 \end{bmatrix}, \quad \mathbf{E}_2 = \begin{bmatrix} 0 & -\frac{Rr_C}{R+r_C} \\ 0 & 0 \end{bmatrix}$$

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State-Space Average Modeling: Averaging

Step 3. Average each state by using duty ratio as a weighting factor and then combine the two sets of equations into a single set.

$$Q:\ \begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}_1\mathbf{x} + \mathbf{B}_1\mathbf{u} \\ \mathbf{y} &= \mathbf{C}_1\mathbf{x} + \mathbf{E}_1\mathbf{u} \end{aligned} \quad \times d_1 \quad + \quad D:\ \begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}_2\mathbf{x} + \mathbf{B}_2\mathbf{u} \\ \mathbf{y} &= \mathbf{C}_2\mathbf{x} + \mathbf{E}_2\mathbf{u} \end{aligned} \quad \times d_2$$

averaging by using state duty ratio weighting

⇩

$$\dot{\mathbf{x}} = (\mathbf{A}_1 d_1 + \mathbf{A}_2 d_2)\mathbf{x} + (\mathbf{B}_1 d_1 + \mathbf{B}_2 d_2)\mathbf{u}$$
$$\mathbf{y} = (\mathbf{C}_1 d_1 + \mathbf{C}_2 d_2)\mathbf{x} + (\mathbf{E}_1 d_1 + \mathbf{E}_2 d_2)\mathbf{u}$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{E}\mathbf{u}$$

where

$$\mathbf{A} = \mathbf{A}_1 d_1 + \mathbf{A}_2 d_2$$
$$\mathbf{B} = \mathbf{B}_1 d_1 + \mathbf{B}_2 d_2$$
$$\mathbf{C} = \mathbf{C}_1 d_1 + \mathbf{C}_2 d_2$$
$$\mathbf{E} = \mathbf{E}_1 d_1 + \mathbf{E}_2 d_2$$

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State-Space Average Modeling: Perturbation

Step 4. Perturb the averaged equation set to produce DC and small signal terms and eliminate nonlinear product terms.

Substitute $\mathbf{x} = \mathbf{X} + \hat{\mathbf{x}}$; $\mathbf{y} = \mathbf{Y} + \hat{\mathbf{y}}$; $\mathbf{u} = \mathbf{U} + \hat{\mathbf{u}}$;

$d_1 = D_1 + \hat{d}$; $d_2 = D_2 - \hat{d}$

into the averaged equations (in Steps 3)

$$\dot{\mathbf{x}} = (\mathbf{A}_1 d_1 + \mathbf{A}_2 d_2)\mathbf{x} + (\mathbf{B}_1 d_1 + \mathbf{B}_2 d_2)\mathbf{u}$$
$$\mathbf{y} = (\mathbf{C}_1 d_1 + \mathbf{C}_2 d_2)\mathbf{x} + (\mathbf{E}_1 d_1 + \mathbf{E}_2 d_2)\mathbf{u}$$

$$\begin{aligned}\frac{d}{dt}(\mathbf{X}+\hat{\mathbf{x}}) &= (\mathbf{A}_1(D_1+\hat{d}) + \mathbf{A}_2(D_2-\hat{d}))(\mathbf{X}+\hat{\mathbf{x}}) + \\ &\quad (\mathbf{B}_1(D_1+\hat{d}) + \mathbf{B}_2(D_2-\hat{d}))(\mathbf{U}+\hat{\mathbf{u}}) \\ &= \mathbf{A}_1 D_1 \mathbf{X} + \mathbf{A}_1 \hat{d}\mathbf{X} + \mathbf{A}_1 D_1 \hat{\mathbf{x}} + \mathbf{A}_1 \hat{d}\hat{\mathbf{x}} + \\ &\quad \mathbf{A}_2 D_2 \mathbf{X} - \mathbf{A}_2 \hat{d}\mathbf{X} + \mathbf{A}_2 D_2 \hat{\mathbf{x}} - \mathbf{A}_2 \hat{d}\hat{\mathbf{x}} + \\ &\quad \mathbf{B}_1 D_1 \mathbf{U} + \mathbf{B}_1 \hat{d}\mathbf{U} + \mathbf{B}_1 D_1 \hat{\mathbf{u}} + \mathbf{B}_1 \hat{d}\hat{\mathbf{u}} + \\ &\quad \mathbf{B}_2 D_2 \mathbf{U} - \mathbf{B}_2 \hat{d}\mathbf{U} + \mathbf{B}_2 D_2 \hat{\mathbf{u}} - \mathbf{B}_2 \hat{d}\hat{\mathbf{u}}\end{aligned}$$

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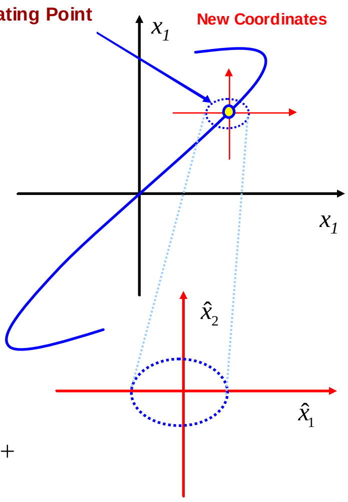
State-Space Average Modeling: Perturbation

Perturbation of the State Equations

$$\begin{aligned}\frac{d}{dt}(\mathbf{X}+\hat{\mathbf{x}}) &= (\mathbf{A}_1 D_1 + \mathbf{A}_2 D_2)\mathbf{X} + (\mathbf{B}_1 D_1 + \mathbf{B}_2 D_2)\mathbf{U} + \\ &\quad (\mathbf{A}_1 D_1 + \mathbf{A}_2 D_2)\hat{\mathbf{x}} + (\mathbf{B}_1 D_1 + \mathbf{B}_2 D_2)\hat{\mathbf{u}} + \\ &\quad [(\mathbf{A}_1 - \mathbf{A}_2)\mathbf{X} + (\mathbf{B}_1 - \mathbf{B}_2)\mathbf{U}]\hat{d} + \\ &\quad (\mathbf{A}_1 - \mathbf{A}_2)\hat{d}\hat{\mathbf{x}} + (\mathbf{B}_1 - \mathbf{B}_2)\hat{d}\hat{\mathbf{u}}\end{aligned}$$

$$\begin{aligned}\mathbf{Y}+\hat{\mathbf{y}} &= (\mathbf{C}_1(D_1+\hat{d}) + \mathbf{C}_2(D_2-\hat{d}))(\mathbf{X}+\hat{\mathbf{x}}) + \\ &\quad (\mathbf{E}_1(D_1+\hat{d}) + \mathbf{E}_2(D_2-\hat{d}))(\mathbf{U}+\hat{\mathbf{u}}) \\ &= (\mathbf{C}_1 D_1 + \mathbf{C}_2 D_2)\mathbf{X} + (\mathbf{E}_1 D_1 + \mathbf{E}_2 D_2)\mathbf{U} + \\ &\quad (\mathbf{C}_1 D_1 + \mathbf{C}_2 D_2)\hat{\mathbf{x}} + [(\mathbf{C}_1 - \mathbf{C}_2)\mathbf{X} + (\mathbf{E}_1 - \mathbf{E}_2)\mathbf{U}]\hat{d} + \\ &\quad (\mathbf{E}_1 D_1 + \mathbf{E}_2 D_2)\hat{\mathbf{u}} + (\mathbf{C}_1 - \mathbf{C}_2)\hat{\mathbf{x}}\hat{d} + (\mathbf{E}_1 - \mathbf{E}_2)\hat{d}\hat{\mathbf{u}}\end{aligned}$$



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State-Space Average Modeling: DC Analysis

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DC Analysis:

Set all variational terms to zero, we can obtain

$$0 = (\mathbf{A}_1 D_1 + \mathbf{A}_2 D_2)\mathbf{X} + (\mathbf{B}_1 D_1 + \mathbf{B}_2 D_2)\mathbf{U}$$
$$= \mathbf{AX} + \mathbf{BU}$$

$$\mathbf{Y} = (\mathbf{C}_1 D_1 + \mathbf{C}_2 D_2)\mathbf{X} + (\mathbf{E}_1 D_1 + \mathbf{E}_2 D_2)\mathbf{U}$$
$$= \mathbf{CX} + \mathbf{EU}$$

Therefore

$$\mathbf{X} = -\mathbf{A}^{-1}\mathbf{BU}$$

$$\mathbf{Y} = (-\mathbf{CA}^{-1}\mathbf{B} + \mathbf{E})\mathbf{U}$$

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State-Space Average Modeling: DC Model

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Eliminate the DC term $\frac{d}{dt}\mathbf{X} = 0$ ➡ $\mathbf{AX} + \mathbf{BU} = 0$ and $\mathbf{Y} = \mathbf{CX} + \mathbf{EU}$

We obtain the dc model equation:

$$D.C. \quad \begin{cases} 0 = [-r_L + (R // r_C)]I_L - \left(\frac{R}{R + r_C}\right)V_C + V_i D \\ 0 = \left(\frac{R}{R + r_C}\right)I_L - \left(\frac{1}{R + r_C}\right)V_C \\ V_o = (R // r_C)I_L - \left(\frac{R}{R + r_C}\right)V_C \\ I_S = DI_L \end{cases}$$

Comments:

1. **DC model gives DC information (steady-state behavior).**
2. **DC model can be used for loss estimation.**

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State-Space Average Modeling: AC Model

Neglect the nonlinear product term $\hat{d}\cdot\hat{\mathbf{x}}$ and $\hat{d}\cdot\hat{\mathbf{u}}$

We obtain the ac model (small-signal) equation:

$$\frac{d}{dt}\hat{\mathbf{x}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\hat{\mathbf{u}} + \mathbf{F}\hat{d}$$

$$\hat{\mathbf{y}} = \mathbf{C}\hat{\mathbf{x}} + \mathbf{E}\hat{\mathbf{u}} + \mathbf{G}\hat{d}$$

where

$$\mathbf{A} = \mathbf{A}_1 D_1 + \mathbf{A}_2 D_2$$
$$\mathbf{B} = \mathbf{B}_1 D_1 + \mathbf{B}_2 D_2$$
$$\mathbf{C} = \mathbf{C}_1 D_1 + \mathbf{C}_2 D_2$$
$$\mathbf{E} = \mathbf{E}_1 D_1 + \mathbf{E}_2 D_2$$
$$\mathbf{F} = (\mathbf{A}_1 - \mathbf{A}_2)\mathbf{X} + (\mathbf{B}_1 - \mathbf{B}_2)\mathbf{U}$$
$$\mathbf{G} = (\mathbf{C}_1 - \mathbf{C}_2)\mathbf{X} + (\mathbf{E}_1 - \mathbf{E}_2)\mathbf{U}$$

$$A.C.\begin{cases} \dfrac{d\hat{i}_L}{dt} = -\left(\dfrac{r_L + R//r_C}{L}\right)\hat{i}_L + \dfrac{1}{L}\left(\dfrac{R}{R+r_C}\right)\hat{v}_C + \left(\dfrac{D}{L}\right)\hat{v}_i + \left(\dfrac{V_i}{L}\right)\hat{d} \\ \dfrac{d\hat{v}_C}{dt} = -\dfrac{1}{C}\left(\dfrac{R}{R+r_C}\right)\hat{i}_L - \dfrac{1}{L}\left(\dfrac{1}{R+r_C}\right)\hat{v}_C \\ \hat{v}_o = (R//r_C)\hat{i}_L + \left(\dfrac{R}{R+r_C}\right)\hat{v}_C \\ \hat{i}_s = D\hat{i}_L + I_L\hat{d} \end{cases}$$

Comments:

1. AC model gives small signal information.

2. AC model parameter value depends on dc operating point.

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Time-to-Frequency Transform

Step 5. Transform ac state-space model into frequency domain (Laplace transform, *s*-domain).

$$\frac{d}{dt}\hat{\mathbf{x}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\hat{\mathbf{u}} + \mathbf{F}\hat{d}$$

$$\hat{\mathbf{y}} = \mathbf{C}\hat{\mathbf{x}} + \mathbf{E}\hat{\mathbf{u}} + \mathbf{G}\hat{d}$$

In the following, the hat above the variables will be neglected for simplicity. The small signal (or ac term) of a variable is denoted using lower case letter. Thus, the dynamic equation of a PWM dc-dc converter can be represented as:

$$\frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{F}d$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{E}\mathbf{u} + \mathbf{G}d$$

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Time-to-Frequency Transform

Take Laplace transform:

$$s\mathbf{x}(s) = \mathbf{A}\mathbf{x}(s) + \mathbf{B}\mathbf{u}(s) + \mathbf{F}d(s)$$

$$\mathbf{y}(s) = \mathbf{C}\mathbf{x}(s) + \mathbf{E}\mathbf{u}(s) + \mathbf{G}d(s)$$

$$\mathbf{x}(s) = (s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\mathbf{u}(s) + (s\mathbf{I}-\mathbf{A})^{-1}\mathbf{F}d(s) = \mathbf{H}_u(s)\mathbf{u}(s) + \mathbf{H}_d(s)d(s)$$

$$\mathbf{H}_u(s) = (s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}$$

$$\mathbf{H}_d(s) = (s\mathbf{I}-\mathbf{A})^{-1}\mathbf{F}d(s)$$

$$\mathbf{y}(s) = [\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{E}]\mathbf{u}(s) + [\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{F}+\mathbf{G}]d(s) = \mathbf{G}_u(s)\mathbf{u}(s) + \mathbf{G}_d(s)d(s)$$

$$\mathbf{G}_u(s) = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{E} \;\Rightarrow\; \mathbf{G}_u(s) = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{E} = \begin{bmatrix} G_{u11}(s) & G_{u12}(s) \\ G_{u21}(s) & G_{u22}(s) \end{bmatrix}$$

$$\mathbf{G}_d(s) = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{F}+\mathbf{G}$$

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State-Space Average Modeling: Transfer Functions

Interpretation of the transfer function matrix

$$\begin{bmatrix} v_o \\ i_i \end{bmatrix} = \begin{bmatrix} G_{u11}(s) & G_{u12}(s) \\ G_{u21}(s) & G_{u22}(s) \end{bmatrix} \begin{bmatrix} v_i \\ i_d \end{bmatrix} + \begin{bmatrix} G_{d1}(s) \\ G_{d2}(s) \end{bmatrix} d$$

- $G_{u11}(s)$: Input-to-output voltage gain (audio susceptibility)
- $G_{u12}(s)$: Output impedance
- $G_{d1}(s)$: Control-to-output voltage gain
- $G_{u21}(s)$: Input Admittance
- $G_{u22}(s)$: Load-to-line current gain
- $G_{d2}(s)$: Control-to-input current gain

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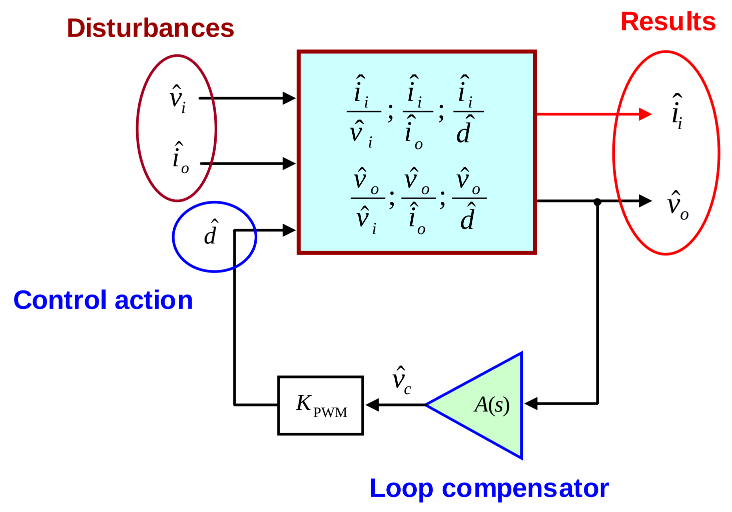
State-Space Average Modeling: Transfer Functions



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State-Space Averaging: Time-to-Frequency Transform

$$\frac{\hat{v}_o(s)}{\hat{d}(s)} = V_i \left[\frac{1 + s r_C C}{1 + s\left(r_C C + [R // r_L]C + \frac{L}{R + r_L} \right) + s^2 LC\left(\frac{R + r_C}{R + r_C + r_L} \right)} \right]$$

$$\frac{\hat{v}_o(s)}{\hat{v}_i(s)} = D \left[\frac{1 + s r_C C}{1 + s\left(r_C C + [R // r_L]C + \frac{L}{R + r_L} \right) + s^2 LC\left(\frac{R + r_C}{R + r_C + r_L} \right)} \right]$$

$$Z_o(s) = \left. \frac{v_o(s)}{i_d(s)} \right|_{\substack{d=0 \\ v_i=0}} = G_{u12}(s)$$

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State-Space Averaging: Time-to-Frequency Transform

Note: In most conditions, because $R >> (r_L + r_C)$, the above equations can be approximated as

$$\frac{\hat{v}_o(s)}{\hat{d}(s)} = V_i\left[\frac{1+sr_CC}{1+s\left(\frac{L}{R}+(r_C+r_L)C\right)+s^2LC}\right] \qquad \frac{\hat{v}_o(s)}{\hat{v}_i(s)} = D\left[\frac{1+sr_CC}{1+s\left(\frac{L}{R}+(r_C+r_L)C\right)+s^2LC}\right]$$

$$\frac{\hat{v}_o(s)}{\hat{d}(s)} = V_i\left[\frac{1+sr_CC}{1+s\left(\frac{L}{R}+(r_C+r_L)C\right)+s^2LC}\right]$$

$$\Delta s = s^2 + \frac{\omega_o}{Q}s + \omega_o^2$$

$$\frac{\hat{v}_o(s)}{\hat{d}(s)} = V_i\frac{1+sr_CC}{\Delta s/\omega_o^2}$$

$$\omega_o = \frac{1}{\sqrt{LC}}$$

$$Q = \frac{1}{\omega_o}\frac{1}{\frac{L}{R}+(r_L+r_C)C}$$

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State-Space Average Modeling

The small signal control-to-output $\frac{\hat{v}_o(s)}{\hat{d}(s)}$, and line-to-output $\frac{\hat{v}_o(s)}{\hat{v}_i(s)}$ are transfer functions of two poles and one zero

$$K\frac{(s+z_1)}{(s+p_1)(s+p_2)}$$

with its parameters depending upon component values and operating point. p_1, p_2 can be complex poles or real poles. But p_1, p_2 and z_1 all lie on LHP.

Note: The equivalent series resistance of the capacitor will introduce a LHP zero.

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State-Space Averaging: Output Impedance

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Output Impedance: $Z_o(s) = \left.\dfrac{v_o(s)}{i_d(s)}\right|_{\substack{d=0\\u=0}} = \mathbf{c}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{du}$

$$\frac{\hat{v}_o(s)}{\hat{i}_d(s)} = R\,\frac{1+\dfrac{s}{\omega_1 Q_1}+\left(\dfrac{s}{\omega_1}\right)^2}{\Delta s/\omega_o^2}$$

$$\Delta s = s^2 + \frac{\omega_o}{Q}s + \omega_o^2$$

$$\omega_o = \frac{1}{\sqrt{LC}}\ , \qquad Q = \frac{1}{\omega_o}\,\frac{1}{\dfrac{L}{R}+(r_L+r_C)C}$$

$$\omega_1 = \frac{1}{\sqrt{LC}}\sqrt{\frac{r_L}{r_C}}\ ; \qquad Q_1 = \frac{1}{\omega_1}\,\frac{1}{\dfrac{L}{r_C}+r_L C}$$

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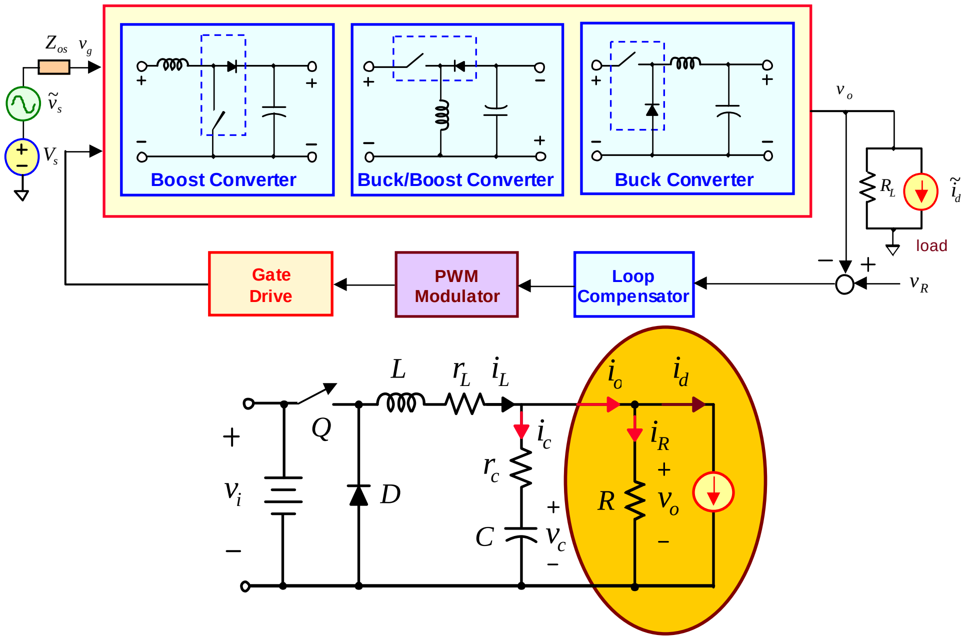
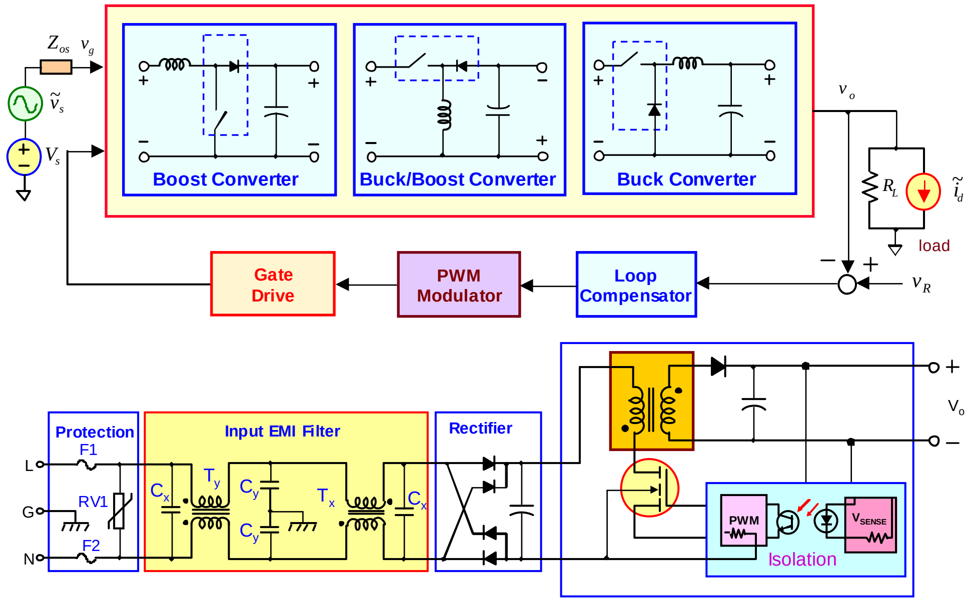
Modeling of Load Impedance and Disturbances

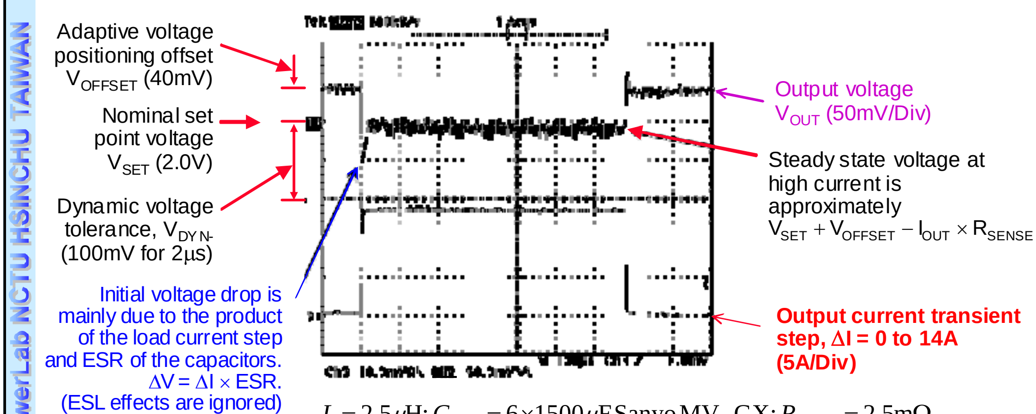


Modeling of Source Impedance and Disturbances



R. D. Middlebrook, "**Input Filter Considerations in Design and Application of Switching Regulators**," IEEE Industry Applications Society Annual Meeting, 1976 Record, pp. 366-382.

Dynamic Responses for Step Load Changes



$L = 2.5\mu H$; $C_{OUT} = 6 \times 1500\mu F$ Sanyo MV-GX; $R_{SENSE} = 2.5m\Omega$

Intel: VRM (Voltage Regulator Module) and Enterprise Voltage Regulator-Down (EVRD) 11.0 Design Guidelines, Nov. 2006.

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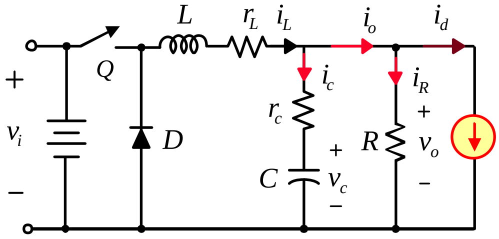
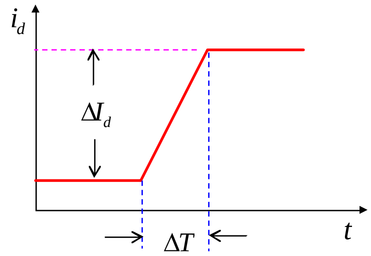
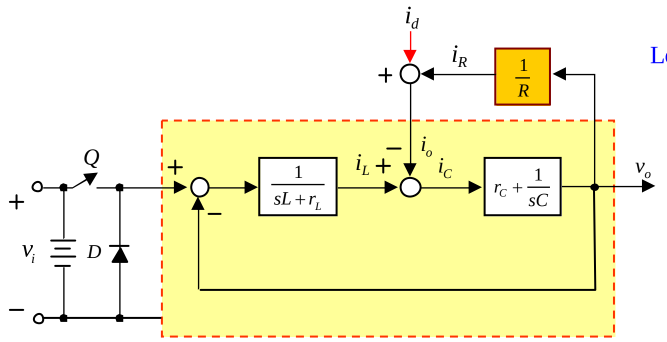
Buck Converter with Load Current Disturbance

Block Diagram Representation



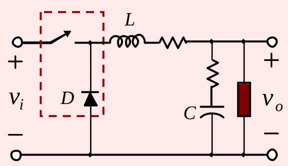


Load Current Slew Rate $\frac{\Delta I_d}{\Delta T}$



Control-to-Output Transfer Functions

DC-DC Converters in CCM Operating Mode

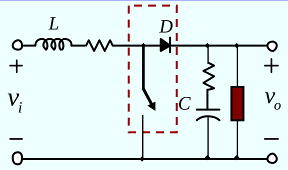


- **Buck**

$$\frac{\hat{v}_o(s)}{\hat{d}(s)} = V_i \frac{1+r_C Cs}{(\frac{s}{\omega_o})^2 + \frac{1}{\omega_o Q}s+1}$$

$$\omega_o = \frac{1}{\sqrt{LC}}$$

$$Q = \frac{1}{\omega_o}\frac{1}{\frac{L}{R}+(r_L+r_C)C}$$

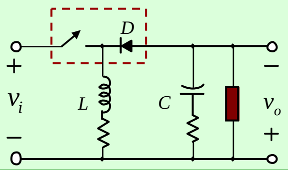


- **Boost**

$$\frac{\hat{v}_o(s)}{\hat{d}(s)} = \frac{V_i}{D^2}\frac{(1+r_C Cs)(1-sL/(D^2R))}{(\frac{s}{\omega_o})^2 + \frac{1}{\omega_o Q}s+1}$$

$$\omega_o = \frac{D}{\sqrt{LC}}$$

$$Q = \frac{D}{\omega_o}\frac{1}{\frac{L}{D\,R}+\frac{r_L C}{D}+r_C C}$$



- **Buck/Boost**

$$\frac{\hat{v}_o(s)}{\hat{d}(s)} = \frac{V_i}{D^2}\frac{(1+r_C Cs)(1-sDL/(D^2R))}{(\frac{s}{\omega_o})^2 + \frac{1}{\omega_o Q}s+1}$$

$$\omega_o = \frac{D}{\sqrt{LC}}$$

$$Q = \frac{D}{\omega_o}\frac{1}{\frac{L}{(D)^2R}+(\frac{r_C}{D}+\frac{r_L}{(D)^2})C}$$

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Transfer Functions of Buck Converters

Transfer Function	Transfer Function
$\frac{\hat{v}_o(s)}{\hat{d}(s)} = V_i \frac{1+r_C Cs}{\Delta s/\omega_o^2} = \frac{F_{D1}}{\Delta}$	$\frac{\hat{i}_L(s)}{\hat{d}(s)} = \frac{V_i}{r_L}\frac{1+r_L Cs}{\Delta s/\omega_o^2} = \frac{F_{D2}}{\Delta}$
$\frac{\hat{v}_o(s)}{\hat{v}_i(s)} = D \frac{1+r_C Cs}{\Delta s/\omega_o^2} = \frac{F_{U11}}{\Delta}$	$\frac{\hat{i}_L(s)}{\hat{v}_i(s)} = \frac{D}{R}\frac{1+RCs}{\Delta s/\omega_o^2} = \frac{F_{U21}}{\Delta}$
$\frac{\hat{v}_o(s)}{\hat{i}_d(s)} = R_{eq}\frac{1+\frac{s}{\omega_1 Q_1}+(\frac{s}{\omega_1})^2}{\Delta s/\omega_o^2} = \frac{F_{U12}}{\Delta}$	$\frac{\hat{i}_L(s)}{\hat{i}_d(s)} = -\frac{1+r_C Cs}{\Delta s/\omega_o^2} = \frac{F_{U22}}{\Delta}$

$$\Delta s = s^2 + \frac{\omega_o}{Q}s + \omega_o^2 , \qquad R_{eq} = r_L$$

$$\omega_o = \frac{1}{\sqrt{LC}} , \qquad Q = \frac{1}{\omega_o}\frac{1}{\frac{L}{R}+(r_L+r_C)C}$$

$$\omega_1 = \frac{1}{\sqrt{LC}}\sqrt{\frac{r_L}{r_C}} \;; \qquad Q_1 = \frac{1}{\omega_1}\frac{1}{\frac{L}{r_C}+r_L C}$$

Transfer Functions of Boost Converters

Transfer Function	Transfer Function
$\frac{\hat{v}_o(s)}{\hat{d}(s)} = \frac{V_i}{(D^{'})^2}\frac{(1+\frac{s}{\omega_z})(1-\frac{s}{\omega_a})}{\Delta s/\omega_o^2} = \frac{F_{D1}}{\Delta}$	$\frac{\hat{i}_L(s)}{\hat{d}(s)} = \frac{2V_I}{(D^{'})^3 R}\frac{1+\frac{RC}{2}}{\Delta s/\omega_o^2} = \frac{F_{D2}}{\Delta}$
$\frac{\hat{v}_o(s)}{\hat{v}_i(s)} = \frac{1}{D^{'}}\frac{1}{\Delta s/\omega_o^2} = \frac{F_{U11}}{\Delta}$	$\frac{\hat{i}_L(s)}{\hat{v}_i(s)} = \frac{2V_i}{(D^{'})^3 R}\frac{1+RC}{\Delta s/\omega_o^2} = \frac{F_{U21}}{\Delta}$
$\frac{\hat{v}_o(s)}{\hat{v}_i(s)} = R_{eq}\frac{1+\frac{s}{\omega_1 Q_1}+(\frac{s}{\omega_1})^2}{\Delta s/\omega_o^2} = \frac{F_{U12}}{\Delta}$	$\frac{\hat{i}_L(s)}{\hat{i}_d(s)} = -\frac{1}{D^{'}}\frac{1+r_C Cs}{\Delta s/\omega_o^2} = \frac{F_{U22}}{\Delta}$

$$\Delta s = s^2 + \frac{\omega_o}{Q}s + \omega_o^2 , \qquad Q = \frac{D^{'}}{\omega_o}\frac{1}{\frac{L}{D^{'}R}+\frac{r_L C}{D^{'}}+D^{'}r_C C}$$

$$\omega_o = \frac{D^{'}}{\sqrt{LC}}$$

$$\omega_1 = \frac{D^{'}}{\sqrt{LC}}\sqrt{\frac{R_{eq}}{r_C}} \;; \qquad Q_1 = \frac{1}{\omega_1}\frac{1}{(\frac{L}{D^{'2}R_{eq}}+r_C C)}$$

$$\omega_z = \frac{1}{r_C C} \;; \; \omega_a = \frac{(D^{'})^2 R}{L} \qquad R_{eq} = \frac{r_L}{D^{'2}}\frac{D}{D^{'}}r_C$$

Transfer Functions of Buck/Boost Converters

Transfer Function	Transfer Function
$\frac{\hat{v}_o(s)}{\hat{d}(s)} = \frac{V_i}{(D')^2} \frac{(1+\frac{s}{\omega_z})(1-\frac{s}{\omega_a})}{\Delta s/\omega_o^2} = \frac{F_{D1}}{\Delta}$	$\frac{\hat{i}_L(s)}{\hat{d}(s)} = \frac{V_i}{(D')^2 R} \frac{1+RCs}{\Delta s/\omega_o^2} = \frac{F_{D2}}{\Delta}$
$\frac{\hat{v}_o(s)}{\hat{v}_i(s)} = \frac{D}{D'} \frac{(1+r_C Cs)}{\Delta s/\omega_o^2} = \frac{F_{U11}}{\Delta}$	$\frac{\hat{i}_L(s)}{\hat{v}_i(s)} = \frac{D}{(D')^2} \frac{1}{R} \frac{1+RCs}{\Delta s/\omega_o^2} = \frac{F_{U21}}{\Delta}$
$\frac{\hat{v}_o(s)}{\hat{i}_d(s)} = R_{eq} \frac{1+\frac{s}{\omega_1 Q_1}+(\frac{s}{\omega_1})^2}{\Delta s/\omega_o^2} = \frac{F_{U12}}{\Delta}$	$\frac{\hat{i}_L(s)}{\hat{i}_d(s)} = -\frac{1}{D'} \frac{1+r_C Cs}{\Delta s/\omega_o^2} = \frac{F_{U22}}{\Delta}$

$$\Delta s = s^2 + \frac{\omega_o}{Q}s + \omega_o^2$$

$$\omega_o = \frac{D'}{\sqrt{LC}} \;, \qquad Q = \frac{D'}{\omega_o} \frac{1}{\frac{L}{(D')^2 R} + (\frac{r_C}{D'} + \frac{r_L}{(D')^2})C}$$

$$\omega_z = \frac{1}{r_C C} \;, \quad \omega_a = \frac{(D')^2 R}{DL} \;; \qquad R_{eq} = \frac{r_L}{(D')^2} + \frac{D}{D'} r_C$$

$$\omega_1 = \omega_o \sqrt{\frac{R_{eq}}{r_C}} \;; \qquad Q_1 = \frac{1}{\omega_1} \frac{1}{\frac{L}{D'^2 R_{eq}} + r_C C}$$

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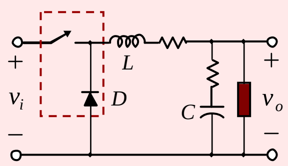
Q-factor of Buck Converter



- **Buck**

$$\frac{\hat{v}_o(s)}{\hat{d}(s)} = V_i \frac{1+r_C Cs}{(\frac{s}{\omega_o})^2 + \frac{1}{\omega_o Q}s + 1}$$

$$\omega_o = \frac{1}{\sqrt{LC}}$$

$$Q = \frac{1}{\omega_o} \frac{1}{\frac{L}{R} + (r_L + r_C)C}$$

$$Q = \frac{1}{\omega_o} \frac{1}{\frac{L}{R} + (r_L + r_C)C} = \frac{1}{\frac{1}{\sqrt{LC}}} \frac{1}{\frac{L}{R} + (r_L + r_C)C} = \frac{1}{\frac{1}{R}\sqrt{\frac{L}{C}} + (r_L + r_C)\sqrt{\frac{C}{L}}} = \frac{1}{\frac{Z_o}{R} + \frac{r_L + r_C}{Z_o}}$$

$$Q = \frac{1}{\omega_o} \frac{1}{\frac{L}{R} + (r_L + r_C)C} = \frac{1}{\frac{Z_o}{R} + \frac{r_L + r_C}{Z_o}}$$

$$Q = \frac{1}{\omega_o} \frac{1}{\frac{L}{R} + (r_L + r_C)C} \approx \frac{\sqrt{LC}}{\frac{L}{R}} = \frac{R}{\sqrt{\frac{L}{C}}}$$

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Summary: Buck Converters at CCM

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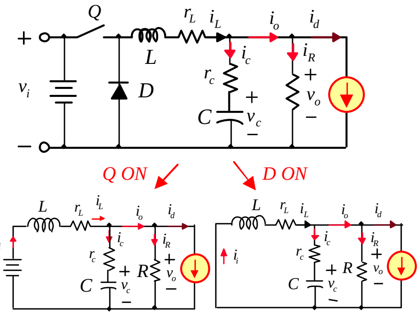
Transfer Function	Transfer Function
$\frac{\hat{v}_o(s)}{\hat{d}(s)} = V_i \frac{1 + r_C Cs}{\Delta s/\omega_o^2} = \frac{F_{D1}}{\Delta}$	$\frac{\hat{i}_L(s)}{\hat{d}(s)} = \frac{V_i}{r_L} \frac{1 + r_L Cs}{\Delta s/\omega_o^2} = \frac{F_{D2}}{\Delta}$
$\frac{\hat{v}_o(s)}{\hat{v}_i(s)} = D \frac{1 + r_C Cs}{\Delta s/\omega_o^2} = \frac{F_{U11}}{\Delta}$	$\frac{\hat{i}_L(s)}{\hat{v}_i(s)} = \frac{D}{R} \frac{1 + RCs}{\Delta s/\omega_o^2} = \frac{F_{U21}}{\Delta}$
$\frac{\hat{v}_o(s)}{\hat{i}_d(s)} = R_{eq} \frac{1 + \frac{s}{\omega_1 Q_1} + (\frac{s}{\omega_1})^2}{\Delta s/\omega_o^2} = \frac{F_{U12}}{\Delta}$	$\frac{\hat{i}_L(s)}{\hat{i}_d(s)} = -\frac{1 + r_C Cs}{\Delta s/\omega_o^2} = \frac{F_{U22}}{\Delta}$



Switch-ON Period

$$\begin{cases}\dot{\mathbf{x}} = \mathbf{A}_1\mathbf{x} + \mathbf{B}_1\mathbf{u} \\ \mathbf{y} = \mathbf{C}_1\mathbf{x} + \mathbf{E}_1\mathbf{u}\end{cases} \times d_1 \; +$$

Switch-OFF Period

$$\begin{cases}\dot{\mathbf{x}} = \mathbf{A}_2\mathbf{x} + \mathbf{B}_2\mathbf{u} \\ \mathbf{y} = \mathbf{C}_2\mathbf{x} + \mathbf{E}_2\mathbf{u}\end{cases} \times d_2$$

averaging by using state duty ratio weighting

$\mathbf{x} = \mathbf{Ax} + \mathbf{Bu}$
$\mathbf{y} = \mathbf{Cx} + \mathbf{Du}$
where
$\mathbf{A} = \mathbf{A}_1 d_1 + \mathbf{A}_2 d_2$
$\mathbf{B} = \mathbf{B}_1 d_1 + \mathbf{B}_2 d_2$
$\mathbf{C} = \mathbf{C}_1 d_1 + \mathbf{C}_2 d_2$
$\mathbf{E} = \mathbf{E}_1 d_1 + \mathbf{E}_2 d_2$

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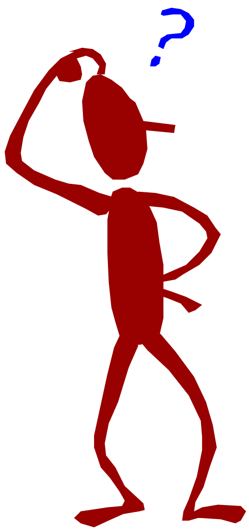

Any Questions ???

Questions inspire effective learning!

學習的關鍵
- 記筆記
- 問問題

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